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Approximation with Monotone Norms in Tensor Product Spaces

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This paper studies approximation problems in spaces of continuous functions with norms different from the usual supremum norm. These norms are assumed to be monotone in many of our results. Questions of proximinality of "blending subspaces" in tensor product settings are investigated. © 1992 Academic Press, Inc.

1. INTRODUCTION

We begin by describing a concrete problem of the type considered in this paper. Continuous functions f, g_i, h_i are prescribed, and we seek continuous functions x_i and y_i to minimize the expression

$$\int_0^1 \int_0^1 \left| f(s, t) - \sum_{j=1}^{m} x_j(s)\, h_j(t) - \sum_{j=1}^{n} y_j(t)\, g_j(s) \right|^p ds\, dt \qquad (1 < p < \infty).$$

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Two features of this problem are noteworthy. First, the problem is set in an incomplete normed linear space, namely, a space of the type $C(S \times T)$ with an L_p-norm. Second, the coefficient functions x_i and y_i that we seek are allowed to vary in infinite-dimensional spaces, namely, $C(S)$ and $C(T)$, where $S = T = [0, 1]$. Because of these considerations it is not clear whether optimal choices of the coefficient functions exist. It is proved below (4.1) that they do.

Many problems similar to the one above can be considered in the much more general setting of monotone or lattice norms. We shall describe such norms on $C(S)$. Let S be a compact Hausdorff space. Then $C(S)$ denotes the space of real-valued continuous functions defined on S. In $C(S)$ the "usual" norm is given by

$$\|x\|_\infty := \sup_{s \in S} |x(s)| \qquad (x \in C(S)).$$

Endowed with this norm, $C(S)$ becomes a Banach space. Now let α be another norm on $C(S)$, written as $\alpha(x)$ or $\|x\|_\alpha$. We say that α is *monotone* if the following implication is valid:

$$0 \leqslant x \leqslant y \Rightarrow \|x\|_\alpha \leqslant \|y\|_\alpha \qquad (x, y \in C(S)).$$

In general, the space $C(S)$ with norm α is incomplete; its completion will be denoted by $C_\alpha(S)$. By this construction, many familiar spaces are obtained, for example, $L_p(S)$ for $1 \leqslant p < \infty$. A monotone norm for which the conditions $0 \leqslant x \leqslant y$ and $\|x\|_\alpha = \|y\|_\alpha$ imply $x = y$ is said to be *strictly monotone.* A monotone norm α for which

$$\| \, |x| \, \|_\alpha = \|x\|_\alpha \qquad (x \in C(S))$$

is called a *lattice norm.*

Each element of $C(S)$ can be decomposed into two parts, x^+ and x^-, such that $0 \leqslant x^+ \leqslant |x|$, $0 \leqslant x^- \leqslant |x|$, and $x = x^+ - x^-$. If the norm α is monotone, then from the inequality

$$0 \leqslant x^+ \leqslant |x| \leqslant \|x\|_\infty = \|x\|_\infty \, 1$$

we obtain $\|x^+\|_\alpha \leqslant \|x\|_\infty \, \|1\|_\alpha$. A similar inequality holds for x^-. Furthermore,

$$\|x\|_\alpha \leqslant \|x^+\|_\alpha + \|x^-\|_\alpha \leqslant 2 \, \|x\|_\infty \, \|1\|_\alpha = \|x\|_\infty \, \|2\|_\alpha.$$

Thus the norm α is topologically weaker than the supremum norm. We now give an example which will illustrate the points made above. On $\mathbb{R}^2$ define a norm α by writing

$$\|(t, s)\|_\alpha = \max\{|t|, |s|, |t - s|\}.$$

Then α is a monotone norm but not a lattice norm. For example, $\|(-1, 1)\|_\alpha = 2$ while $\| |(-1, 1)| \|_\alpha = \|(1, 1)\|_\alpha = 1$. This computation also shows that the inequality

$$\|x\|_\alpha \leqslant \| |x| \|_\alpha$$

may be false for monotone norms. We also observe from the equality

$$2 = \|(-1, 1)\|_\alpha = 2 \|(1, 1)\|_\alpha = 2 \|(1, 1)\|_\infty = \|(2, 2)\|_\alpha \|(-1, 1)\|_\infty$$

that in bounding the α-norm by the ∞-norm the constant $\|2\|_\alpha$ is the best possible one.

If α is a monotone norm on $C(S)$, then as we have seen, α is topologically weaker than the the uniform norm. Thus the identity map $i: (C(S), \| \|_\infty) \to (C(S), \alpha)$ is continuous. It extends to a continuous map $i: C(S) \to C_\alpha(S)$. This map is injective because α is a genuine norm (not just a pseudonorm) on $C(S)$. Thus i qualifies as an *embedding* of $C(S)$ into $C_\alpha(S)$. An element of $C_\alpha(S)$ is continuous if it is in the range of i.

If Y is a normed linear space then $C(S, Y)$ will denote the space of all continuous maps $f: S \to Y$, normed by defining

$$|||f|||_\infty := \sup_{s \in S} \|f(s)\|_Y.$$

If Y is a Banach space, then with this norm $C(S, Y)$ is complete.

If a monotone norm α is given on $C(S)$ we "lift" α to $C(S, Y)$ by defining

$$|||f|||_\alpha := \|Jf\|_\alpha, \qquad (Jf)(s) := \|f(s)\|_Y, \qquad f \in C(S, Y).$$

In this equation $Jf \in C(S)$ and J is a mapping from $C(S, Y)$ to $C(S)$. The following properties of J are easily seen:

(i) The mapping J is nonlinear and norm-preserving;

(ii) If α is a lattice norm, then $\|Jf - Jg\|_\alpha \leqslant |||f - g|||_\alpha$ for $f, g \in C(S, Y)$;

(iii) $J(f + g) \leqslant Jf + Jg$ for $f, g \in C(S, Y)$.

The proof that α is a norm on $C(S, Y)$ is elementary, the monotonicity of α being required for the triangle inequality. Observe that when $Y = \mathbb{R}$, the "lifted" norm α on $C(S, \mathbb{R})$ is not necessarily consistent with the norm α on $C(S)$ since if $f \in C(S, \mathbb{R})$ we have

$$|||f|||_\alpha = \|Jf\|_\alpha = \| |f| \|_\alpha.$$

It does not follow that $\| |f| \|_\alpha = \|f\|_\alpha$. If we want this equality to hold we must assume that α is a lattice norm. Note further that even if Y is a Banach space, $[C(S, Y), \alpha]$ is not in general complete.

In order that there be no confusion as to which norm is intended when a topological or metrical notion is introduced, we use the name of the norm as a prefix. Thus, for example, an α-proximity map of $C(S)$ onto a subspace G is a map $A\colon C(S) \to G$ such that

$$\|x - Ax\|_\alpha \leqslant \|x - g\|_\alpha \qquad (x \in C(S),\ g \in G).$$

Whenever such a map exists we say that G is a *α-proximinal* in $C(S)$.

In the next section we investigate proximinality in $[C(S, Y), \alpha]$ and explore briefly the geometric structure of this space. In Section 3 we show how the lifted α-norm on $C(S, Y)$ may be used in a natural way to define a norm on the tensor product $[C(S), \alpha] \otimes Y$. This section also contains several results about the density of one space in another, in addition to a brief description of tensor products. In Section 4 we discusss the space $[C(S), \alpha] \otimes [C(T), \beta]$, where T is also a compact Hausdorff space and β is another monotone or lattice norm. Thus the general normed linear space Y has been replaced here by $[C(T), \beta]$. Again, the interest here is in proximinality, and we have already mentioned at the outset a consequence of the results of this section.

2. Proximinality in $C(S, Y)$

2.1. Theorem. *Let α be a monotone norm on $C(S)$. Let B be a continuous proximity map of a normed space Y onto a subspace $H \subset Y$. For $f \in C(S, Y)$ define $\bar{B}f := B \circ f$. Then $\bar{B}$ is an ∞-continuous α-proximity map of $C(S, Y)$ onto $C(S, H)$. In particular, $C(S, H)$ is α-proximinal.*

Proof. It is clear that $\bar{B}f \in C(S, H)$. If $g \in C(S, H)$ then the properties of B yield, for each $s \in S$,

$$\|f(s) - B(f(s))\| \leqslant \|f(s) - g(s)\|.$$

In terms of the map J previously defined, this inequality states that

$$0 \leqslant J(f - B \circ f) \leqslant J(f - g).$$

Since α is a monotone norm on $C(S)$, we infer that

$$\|J(f - B \circ f)\|_\alpha \leqslant \|J(f - g)\|_\alpha.$$

By the definition of α on $C(S, Y)$ we have

$$|||f - B \circ f|||_\alpha \leqslant |||f - g|||_\alpha.$$

Thus $\bar{B}$ is an α-proximity map. Its continuity in the ∞-norm follows from 11.8 in [LC] and the observation that the argument given there does not depend on Y being a Banach space. ∎

2.2. COROLLARY. *Let α be a monotone norm on $C(S)$. Let H be a Chebyshev subspace in a normed linear space Y. If Y is an E-space or if H is finite-dimensional, then there is an ∞-continuous α-proximity map of $C(S, Y)$ onto $C(S, H)$.*

Proof. By a theorem in [H, p. 164], either of the hypotheses on Y or H is sufficient to ensure the existence of a continuous proximity map of Y onto H. The preceding theorem is then applied to complete the proof. ∎

In both of the above results it is worthwhile observing that no conclusion can be drawn about the α-continuity of the maps in question. This differs from the case when A is an α-proximity map from $C(S)$ onto G and G is finite-dimensional. The equivalence of norms on G and the fact that α is topologically weaker than ∞ on the domain of A combine to make ∞-continuity a weaker property than α-continuity.

2.3. COROLLARY. *If S and T are intervals in $\mathbb{R}$ and if $\{g_1, ..., g_n\}$ is a Chebyshev system in $C(T)$, then for each $f \in C(S \times T)$ the following infimum is attained*:

$$\inf_{x_i \in C(S)} \sup_{s \in S} \int_T \left| f(s, t) - \sum_{j=1}^{n} x_j(s)\, g_j(t) \right| dt.$$

Proof. By a classical result of Jackson, the space spanned by the functions g_i has the Chebyshev property with respect to the L_1-norm in $C(T)$. Hence the preceding corollary is applicable. ∎

The following useful result was pointed out by an anonymous referee.

2.4. PROPOSITION. *If H is a subspace of a normed space Y and if $\varepsilon > 0$, then there is a continuous map p: $Y \to H$ such that $\|y - p(y)\| < d(y, H) + \varepsilon$ for all $y \in Y$.*

Proof. Define the set-valued map

$$P(y) = \{h \in H : \|y - h\| < d(y, H) + \varepsilon\}.$$

The map P is lower semicontinuous, and its values are nonempty convex subsets of H. An application of Lemma 4.1 in Michael's paper [M] completes the proof. Michael's lemma can also be found in [H2, p. 182]. ∎

We shall now derive a formula for α-distances from $f \in C(S, Y)$ to subspaces of the form $C(S, H)$. This formula encompasses 2.10 of [LC] although the proof given here is substantially different.

2.5. THEOREM. *Let α be a monotone norm on $C(S)$, and let H be a subspace in a normed linear space Y. For $f \in C(S, Y)$ define*

$$F(s) = \operatorname{dist}(f(s), H).$$

Then

$$\operatorname{dist}_\alpha(f, C(S, H)) = \|F\|_\alpha.$$

Proof. If $g \in C(S, H)$ then

$$[J(f-g)](s) = \|f(s) - g(s)\| \geqslant \operatorname{dist}(f(s), H) = F(s).$$

By the monotonicity of α,

$$|||f - g|||_\alpha = \|J(f-g)\|_\alpha \geqslant \|F\|_\alpha.$$

By taking an infimum, we get

$$\operatorname{dist}_\alpha(f, C(S, H)) \geqslant \|F\|_\alpha.$$

For the reverse inequality we use the preceding proposition. Use the map p given above and let $g = p \circ f$. An elementary calculation shows that

$$\operatorname{dist}_\alpha(f, C(S, H)) \leqslant |||f - g|||_\alpha \leqslant \|F\|_\alpha + \varepsilon \, \|1\|_\alpha. \quad \blacksquare$$

The next result is a characterization theorem for best approximations of arbitrary functions in $C(S, Y)$ by elements of the subspace $C(S, H)$ when a strictly monotone norm is used. By using translations it suffices to address the case of a function having 0 as a best approximation in $C(S, H)$.

2.6. PROPOSITION. *Let H be a subspace in a normed space Y, and let α be a strictly monotone norm on $C(S)$. For an f in $C(S, Y)$ these properties are equivalent*:

(i) $|||f|||_\alpha = \operatorname{dist}_\alpha(f, C(S, H))$.

(ii) $\|f(s)\| = \operatorname{dist}(f(s), H)$ *for each* $s \in S$.

Proof. Using the function F in 2.5, we note that $Jf \geqslant F \geqslant 0$ since

$$(JF)(s) = \|f(s)\| \geqslant \operatorname{dist}(f(s), H) = F(s) \geqslant 0.$$

The monotonicity of α and 2.5 yield

$$|||f|||_\alpha = \|Jf\|_\alpha \geqslant \|F\|_\alpha = \operatorname{dist}_\alpha(f, C(S, H)).$$

If (i) is true, the previous inequality becomes an equality, and by the strict monotonicity of α, (ii) follows. If (ii) is true then $Jf=F$ and (i) follows. ∎

We conclude this section with a very natural result about the strict convexity of the space $[C(S, Y), \alpha]$. We need first an elementary result.

2.7. LEMMA. *A strictly convex monotone norm on $C(S)$ is strictly monotone.*

Proof. Let α be a strictly convex, monotone norm. If $0 \leqslant x \leqslant y$ and $\|x\|_\alpha = \|y\|_\alpha$, then

$$0 \leqslant 2x \leqslant x+y \leqslant 2y.$$

Since α is a monotone norm,

$$\|x\|_\alpha + \|y\|_\alpha = 2\,\|x\|_\alpha \leqslant \|x+y\|_\alpha \leqslant 2\,\|y\|_\alpha = \|x\|_\alpha + \|y\|_\alpha.$$

By the strict convexity of α, we infer that $x=y$. ∎

A consequence of the above lemma is that the L_p-norms $(1<p<\infty)$ are strictly monotone. However, there are norms which are strictly monotone but not strictly convex, an example being the L_1-norm.

2.8. THEOREM. *Let α be a lattice norm on $C(S)$, and let Y be a normed linear space. In order that $C(S, Y)$ be strictly convex with the α-norm, it is necessary and sufficient that α and Y be strictly convex.*

Proof. For the sufficiency, let f and g be elements of $C(S, Y)$ such that

$$|||f|||_\alpha = |||g|||_\alpha = \tfrac{1}{2}\,|||f+g|||_\alpha = 1.$$

In terms of the mapping J this yields

$$\begin{aligned} 1 = \|Jf\|_\alpha = \|Jg\|_\alpha &= \tfrac{1}{2}\,\|J(f+g)\|_\alpha \\ &\leqslant \tfrac{1}{2}\,\|Jf+Jg\|_\alpha \leqslant \tfrac{1}{2}\,\|Jf\|_\alpha + \tfrac{1}{2}\,\|Jg\|_\alpha = 1. \end{aligned}$$

By the strict convexity of α, it follows that $Jf=Jg$. Now observe that

$$0 \leqslant J(f+g) \leqslant Jf+Jg \qquad \text{and} \qquad \|J(f+g)\|_\alpha = \|Jf+Jg\|_\alpha.$$

By 2.7, we conclude that $J(f+g)=Jf+Jg$. This in turn implies that for all $s \in S$,

$$\|f(s)+g(s)\| = \|f(s)\| + \|g(s)\|.$$

Since $Jf=Jg=\|f(s)\|=\|g(s)\|$, the strict convexity of Y now implies that $f(s)=g(s)$.

For the necessity of the conditions, suppose that Y is not strictly convex. Then there exist distinct elements y_1 and y_2 in Y, with $\|y_1\| = \|y_2\| =$

$\frac{1}{2}\|y_1+y_2\|$. Define f_i in $C(S, Y)$ by putting $f_i(s)=y_i$ $(i=1,2)$ for all $s\in S$. Then $|||f_1|||_\alpha=|||f_2|||_\alpha=\frac{1}{2}|||f_1+f_2|||_\alpha$, and thus $C(S, Y)$ is not strictly convex.

If α is not strictly convex then there exist distinct functions x_1 and x_2 in $C(S)$ such that $\|x_1\|_\alpha=\|x_2\|_\alpha=\frac{1}{2}\|x_1+x_2\|_\alpha$. Select a nonzero element $y\in Y$, and put $f_i(s)=x_i(s)\,y$ for $i=1, 2$. Then in $C(S, Y)$ we have $|||f_1|||_\alpha=|||f_2|||_\alpha=\frac{1}{2}|||f_1+f_2|||_\alpha$, so that $C(S, Y)$ is again not strictly convex. Note that except for this last paragraph, the proof is valid for a monotone norm. ∎

2.9. Proposition. *Let μ be a Borel measure on S which assigns positive measure to each nonvoid open set. Assume that $\mu(S)=1$, and put $\beta(x)=\int|x(s)|\,d\mu(s)$. Then β is a lattice norm and is minimal among the lattice norms which satisfy $\alpha(1)=1$.*

Proof. Let α be a lattice norm satisfying $\alpha(1)=1$ and $\alpha\leqslant\beta$. We shall prove that $\alpha=\beta$. If this equality is not true, there exists $z\in C(S)$ for which $\alpha(z)\neq\beta(z)$. Clearly $z\neq 0$, and so we can assume $\beta(z)=1$. Since $\alpha\leqslant\beta$, we have $\alpha(z)<1=\beta(z)$. Since α and β are lattice norms, we can assume that $z\geqslant 0$. Since $\alpha(z)<1$, $z\neq 1$. Since $\beta(1)=\beta(z)$, the inequality $z\leqslant 1$ cannot be true, for it would imply $z=1$. Hence $\|z\|_\infty>1$. Put $\theta=(\|z\|_\infty-1)/\|z\|_\infty$, and define u by the equation $1=\theta u+(1-\theta)z$. Then, because $0<\theta<1$, we have

$$\theta u=1-(1-\theta)z\geqslant 1-(1-\theta)\,\|z\|_\infty=1-\|z\|_\infty+\theta\,\|z\|_\infty=0.$$

Hence $u\geqslant 0$. From the additivity of β on the positive elements,

$$1=\beta(1)=\theta\beta(u)+(1-\theta)\,\beta(z)=\theta\beta(u)+1-\theta.$$

This shows that $\beta(u)=1$. On the other hand

$$1=\alpha(1)=\alpha[\theta u+(1-\theta)z]\leqslant\theta\alpha(u)+(1-\theta)\,\alpha(z)<\theta\alpha(u)+(1-\theta).$$

This shows that $\alpha(u)>1$. Hence $\alpha(u)>\beta(u)$. This contradicts the assumption that $\alpha\leqslant\beta$, and concludes the proof. ∎

3. Some Results in Tensor Product Theory

If X and Y are normed linear spaces then the expression $\sum_{i=1}^n x_i\otimes y_i$, where $x_i\in X$, $y_i\in Y$, and $n\in\mathbb{N}$, is interpreted as an element of $\mathscr{L}(X^*, Y)$ by writing

$$\left(\sum_{i=1}^n x_i\otimes y_i\right)(\phi)=\sum_{i=1}^n \phi(x_i)\,y_i \qquad (\phi\in X^*).$$

Two expressions are regarded as equivalent if they define the same element in $\mathscr{L}(X^*, Y)$. Then $X \otimes Y$ is the set of all equivalence classes of such expressions and forms a linear space when the algebraic notions in $X \otimes Y$ are derived from the operator interpretation. A norm ω on $X \otimes Y$ must give the same value for equivalent expressions. If ω has the property

$$\omega(x \otimes y) = \|x\| \, \|y\| \qquad (x \in X,\ y \in Y)$$

then ω is said to be a *crossnorm*. We say that ω is a *reasonable norm* if, for all $\phi \in X^*$ and $\psi \in Y^*$, the linear form $\phi \otimes \psi$ is bounded on $[X \otimes Y, \omega]$ and has norm equal to $\|\phi\| \, \|\psi\|$. In order that ω be a reasonable crossnorm it is sufficient to have $\omega(x \otimes y) \leqslant \|x\| \, \|y\|$ for all $x \in X$, $y \in Y$, and $\|\phi \otimes \psi\| \leqslant \|\phi\| \, \|\psi\|$ for all $\phi \in X^*$, $\psi \in Y^*$. (See [DU] or [LC], p. 4].) The completion of the normed linear space $[X \otimes Y, \omega]$ is denoted by $X \otimes_\omega Y$.

An important example of a reasonable crossnorm is obtained by assigning to each member of $X \otimes Y$ the norm it has when regarded as an operator from X^* to Y. The resulting norm is denoted by λ and is defined by

$$\lambda\left(\sum_{i=1}^{n} x_i \otimes y_i\right) := \sup\left\{\left\|\sum_{i=1}^{n} \phi(x_i)\, y_i\right\| : \phi \in X^*,\ \|\phi\| = 1\right\}.$$

In fact, λ is the least of the reasonable crossnorms ([DU] or [LC], p. 5]), so that if ω is also a reasonable crossnorm on $X \otimes Y$ then $\omega \geqslant \lambda$.

Another important crossnorm, γ, is defined by

$$\gamma(z) := \inf \sum \|x_i\| \, \|y_i\|,$$

where the infimum is over all representations, $\sum x_i \otimes y_i$, of z. If ω is any crossnorm on $X \otimes Y$ then $\omega \leqslant \gamma$.

It is known ([DU] or [LC, p. 9]) that $C(S) \otimes_\lambda Y = C(S, Y)$ for any Banach space Y and any compact Hausdorff space S. The isometry here is defined by

$$\sum_{i=1}^{n} x_i \otimes y_i \mapsto f, \qquad \text{where} \quad f(s) = \sum_{i=1}^{n} x_i(s)\, y_i.$$

If a crossnorm ω is defined on $X \otimes Y$ and if U and V are subspaces of X and Y, respectively, then ω is well-defined (by restriction) on $U \otimes V$. It is tacitly understood throughout this paper that $U \otimes V$ is so normed and that $U \otimes_\omega V$ is the ω-closure of $U \otimes V$ in $X \otimes_\omega Y$. Our usage on this matter differs from that common in the general theory of tensor products but avoids the difficulty which might otherwise arise if we were to regard ω as defined solely on $U \otimes V$ without reference to the fact that $U \otimes V$ is a subspace of $X \otimes Y$.

We begin by establishing some rather technical results which will be used in the remainder of this paper. The first of these has a straightforward proof, which we omit.

3.1. LEMMA. *If U and V are dense subspaces in normed spaces X and Y, respectively, then every bilinear functional of norm* 1 *on $U \times V$ has an extension to a bilinear functional of norm* 1 *on $X \times Y$.*

3.2. LEMMA. *Under the hypotheses of* 3.1, *the crossnorm γ on $U \otimes V$ is the restriction to $U \otimes V$ of the crossnorm γ on $X \otimes Y$.*

Proof. The norm γ on $U \otimes V$ and on $X \otimes Y$ will be denoted by $\gamma_{U \otimes V}$ and $\gamma_{X \otimes Y}$, respectively. Let w be an element of $U \otimes V$. The definition of γ shows that

$$\gamma_{X \otimes Y}(w) \leqslant \gamma_{U \otimes V}(w).$$

For the reverse inequality, recall the theorem [DU, p. 226] that

$$\gamma_{U \otimes V}(w) = \sup\{\Phi(w) : \Phi \in \mathscr{B}(U, V), \|\Phi\| = 1\}.$$

Here $\mathscr{B}(U, V)$ denotes the space of continuous bilinear functionals on the Cartesian product $U \times V$. Let $\varepsilon > 0$, and select $\Phi \in \mathscr{B}(U, V)$ so that $\|\Phi\| = 1$ and

$$\Phi(w) \geqslant \gamma_{U \otimes V}(w) - \varepsilon.$$

By the preceding lemma, Φ has an extension $\Phi' \in \mathscr{B}(X, Y)$ with $\|\Phi'\| = 1$. Hence

$$\gamma_{X \otimes Y}(w) \geqslant \Phi'(w) = \Phi(w) \geqslant \gamma_{U \otimes V}(w) - \varepsilon.$$

Since ε was arbitrary, this completes the proof. ∎

3.3. LEMMA. *Let U and V be dense subspaces in normed linear spaces X and Y, respectively. Let ω be a reasonable crossnorm on $U \otimes V$. Then there is a unique reasonable crossnorm $\bar{\omega}$ on $X \otimes Y$ that extends ω.*

Proof. Let $z \in X \otimes Y$, and let one of its representations be $z = \sum x_i \otimes y_i$. Select $u_{ki} \in U$ and $v_{ki} \in V$ so that $\|u_{ki} - x_i\| \to 0$ and $\|v_{ki} - y_i\| \to 0$ as $k \to \infty$. If $\bar{\omega}$ is a crossnorm on $X \otimes Y$ that extends ω, then we must have

$$\left|\bar{\omega}\left(\sum x_i \otimes y_i\right) - \bar{\omega}\left(\sum u_{ki} \otimes v_{ki}\right)\right|$$

$$\leqslant \bar{\omega}\left[\sum x_i \otimes y_i - \sum u_{ki} \otimes v_{ki}\right]$$

$$\leqslant \bar{\omega}\left[\sum (x_i - u_{ki}) \otimes y_i\right] + \bar{\omega}\left[\sum u_{ki} \otimes (y_i - v_{ki})\right]$$

$$\leqslant \sum \bar{\omega}[(x_i - u_{ki}) \otimes y_i] + \sum \bar{\omega}[u_{ki} \otimes (y_i - v_{ki})]$$

$$= \sum \|x_i - u_{ki}\| \, \|y_i\| + \sum \|u_{ki}\| \, \|y_i - v_{ki}\|.$$

This shows that $\bar{\omega}$ must be defined by the equation

$$\bar{\omega}(z) = \bar{\omega}\left(\sum x_i \otimes y_i\right) = \lim_{k \to \infty} \omega\left(\sum u_{ki} \otimes v_{ki}\right)$$

Therefore we adopt this equation as the definition of $\bar{\omega}$. Put $z_k = \sum u_{ki} \otimes v_{ki}$. It must be proved that $\lim \omega(z_k)$ exists. By a calculation similar to the previous one, we obtain

$$|\omega(z_k) - \omega(z_m)| \leqslant \omega(z_k - z_m) \leqslant \sum \|u_{ki} - u_{mi}\| \, \|v_{ki}\| + \sum \|u_{mi}\| \, \|v_{ki} - v_{mi}\|. \tag{1}$$

This establishes the Cauchy property of the sequence $\omega(z_k)$.

Next, we show that the definition of $\bar{\omega}(z)$ is independent of the representation of z and independent of the sequences $[u_{ki}]$, $[v_{ki}]$ in the definition. To this end, let $z = \sum x_i \otimes y_i = \sum x'_i \otimes y'_i$. Select sequences $u_{ki} \to x_i$, $v_{ki} \to y_i$, $u'_{ki} \to x'_i$, and $v'_{ki} \to y'_i$ as before. Put $z_k = \sum u_{ki} \otimes v_{ki}$ and $z'_k = \sum u'_{ki} \otimes v'_{ki}$. Let γ_1 and γ_2 be the greatest crossnorms on $U \otimes V$ and $X \otimes Y$, respectively. By the definition of γ_2, and by the same sort of calculation used previously,

$$\gamma_2(z - z_k) = \gamma_2\left(\sum x_i \otimes y_i - \sum u_{ki} \otimes v_{ki}\right)$$

$$\leqslant \sum \|x_i - u_{ki}\| \, \|y_i\| + \sum \|u_{ki}\| \, \|y_i - v_{ki}\| \to 0.$$

Similarly, $\gamma_2(z - z'_k) \to 0$. Hence $\gamma_2(z_k - z'_k) \to 0$. By 3.2 and by the greatest-crossnorm property of γ_1 in $U \otimes V$,

$$\omega(z_k - z'_k) \leqslant \gamma_1(z_k - z'_k) = \gamma_2(z_k - z'_k) \to 0.$$

Next we show that $\bar{\omega}$ is a genuine norm on $X \otimes Y$. Let z be a nonzero element of $X \otimes Y$. Using the notation and arguments above and the fact that ω is reasonable, we have

$$0 < \lambda(z) = \lim \lambda(z_k) \leqslant \lim \omega(z_k) = \bar{\omega}(z).$$

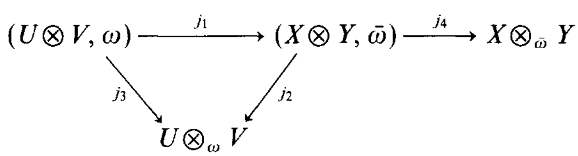
To see that $\bar{\omega}$ is reasonable, it suffices to write

$$|(\phi \otimes \psi)(z)| = \lim |(\phi \otimes \psi)(z_k)| \leqslant \lim \|\phi\| \; \|\psi\| \; \omega(z_k) = \|\phi\| \; \|\psi\| \; \bar{\omega}(z).$$

To complete the proof, we show that $\bar{\omega}$ is a crossnorm. Given $x \otimes y \in X \otimes Y$, select $u_k \in U$ and $v_k \in V$ so that $u_k \to x$ and $v_k \to y$. Then the preceding parts of the proof yield

$$\bar{\omega}(x \otimes y) = \lim \omega(u_k \otimes v_k) = \lim \|u_k\| \; \|v_k\| = \|x\| \; \|y\|. \quad \blacksquare$$

At this juncture it may be helpful to introduce a diagram.



In this diagram, j_3 and j_4 are the natural embeddings of normed linear spaces into their completions; j_1 is the inclusion map, and j_2 is the subject of the next lemma.

3.4. Lemma. *Let U, V, X, Y, ω, and $\bar{\omega}$ be as in* 3.3. *Then there is a linear, norm-preserving map $j_2\colon (X \otimes Y, \bar{\omega}) \to U \otimes_\omega V$.*

Proof. If $z \in X \otimes Y$, we let x_i, y_i, u_{ki}, v_{ki}, z_k be as in the preceding proof. Inequality (1) in that proof shows that $[z_k]$ is an ω-Cauchy sequence in $U \otimes V$. We define $j_2(z)$ as the equivalence class containing $[z_k]$. Other arguments in the preceding proof show that the definition of $j_2(z)$ does not depend on the representation of z nor on the sequences $[u_{ki}]$, $[v_{ki}]$.

That j_2 is norm-preserving follows from writing

$$\omega(j_2(z)) = \lim \omega(z_k) = \bar{\omega}(z).$$

The linearity of j_2 is elementary. $\blacksquare$

3.5. Lemma. *Let U, V, X, Y, ω, $\bar{\omega}$ be as in* 3.3. *Then $j_1(U \otimes V)$ is dense in $(X \otimes Y, \bar{\omega})$.*

Proof. Since j_2 is an embedding, it suffices to prove that $j_2 j_1(U \otimes V)$ is dense in $j_2(X \otimes Y)$. This density follows from the inclusions

$$j_3(U \otimes V) \subset j_2 j_1(U \otimes V) \subset j_2(X \otimes Y) \subset U \otimes_\omega V$$

and from the fact that $j_3(U \otimes V)$ is dense in $U \otimes_\omega V$. ∎

3.6. LEMMA. *Let U, V, X, Y, ω, $\bar{\omega}$ be as before. Then the spaces $X \otimes_{\bar{\omega}} Y$ and $U \otimes_\omega V$ are isometrically isomorphic under the natural map.*

Proof. Since $j_4 j_1(U \otimes V, \omega) \subset j_4(X \otimes Y, \bar{\omega}) \subset X \otimes_{\bar{\omega}} Y$ the map $j_4 j_1$ can be uniquely "extended" to an embedding

$$j_5 : U \otimes_\omega V \to X \otimes_{\bar{\omega}} Y.$$

By this we mean that $j_5 j_3 = j_4 j_1$. Similarly, there is an embedding

$$j_6 : X \otimes_{\bar{\omega}} Y \to U \otimes_\omega V$$

such that $j_6 j_4 = j_2$. Now we observe that $j_2 j_1 = j_3$ because for $z \in U \otimes V$, $j_2 j_1(z)$ and $j_3(z)$ are both equal to the Cauchy sequence $[z, z, z, \ldots]$. Next, it is to be shown that $j_5 j_6$ is the identity on $X \otimes_{\bar{\omega}} Y$. Since $j_4 j_1(U \otimes V)$ is dense in $X \otimes_{\bar{\omega}} Y$, it suffices to prove that $j_5 j_6 j_4 j_1 = j_4 j_1$. From our previous work,

$$j_5 j_6 j_4 j_1 = j_5 j_2 j_1 = j_5 j_3 = j_4 j_1. \quad ∎$$

A crossnorm ω on a tensor product $X \otimes Y$ is said to be *uniform* if the inequality

$$\omega\left(\sum A x_i \otimes B y_i\right) \leqslant \|A\| \, \|B\| \, \omega\left(\sum x_i \otimes y_i\right)$$

is valid whenever $x_i \in X$, $y_i \in Y$, $A \in \mathscr{L}(X, X)$, and $B \in \mathscr{L}(Y, Y)$.

In the next lemma, we have a monotone norm α on $C(S)$ and a monotone norm β on $C(T)$. Finite-dimensional subspaces G and H are given in $C(S)$ and $C(T)$, respectively.

3.7. LEMMA. *Let $u \in G \otimes C_\beta(T)$ and $v \in C_\alpha(S) \otimes H$. If $u + v \in G \otimes C(T) + C(S) \otimes H$, then $u \in G \otimes C(T)$ and $v \in C(S) \otimes H$.*

Proof. Since G and H are finite-dimensional, there exist bounded linear projections $P: C(S) \twoheadrightarrow G$ and $Q: C(T) \twoheadrightarrow H$. Let $\bar{P} = P \otimes_\lambda I_1$ and $\bar{Q} = I_2 \otimes_\lambda Q$, where I_1 and I_2 are the identity maps on $C(T)$ and $C(S)$, respectively. By the general theory ([LC, p. 126], for example), $\bar{P}$ and $\bar{Q}$ are projections of $C(S \times T)$ onto $G \otimes C(T)$ and $C(S) \otimes H$, respectively. These projections commute with each other. The Boolean sum $\bar{P} \oplus \bar{Q}$ is a projection of $C(S \times T)$ onto $G \otimes C(T) + C(S) \otimes H$.

Let $w = u + v$. By hypothesis, $w \in G \otimes C(T) + C(S) \otimes H$. Hence $w = (\bar{P} \oplus \bar{Q})w = \bar{P}w + \bar{Q}(w - \bar{P}w) = \bar{u} + \bar{v}$, where $\bar{u} = \bar{P}w$ and $\bar{v} = \bar{Q}(w - \bar{P}w)$. Then $u - \bar{u} = \bar{v} - v \in [G \otimes C_\beta(T)] \cap [C_\alpha(S) \otimes H] = G \otimes H$. Since $\bar{u} \in G \otimes C(T)$ and $u - \bar{u} \in G \otimes H$, we conclude that $u \in G \otimes C(T)$. Similarly $v \in C(S) \otimes H$. ∎

Let Y be a normed linear space and z an element of $C(S) \otimes Y$. Write $z = \sum x_i \otimes y_i$. Then, as mentioned earlier in this section, we can associate the equivalence class containing z uniquely with the element $f_z \in C(S, Y)$ by the equation $f_z(s) = \sum x_i(s)\, y_i$. It is now an easy matter to transfer the "lifted" α-norm from $C(S, Y)$ to $C(S) \otimes Y$ in such a way that it is independent of the representation of z. This is done by putting

$$\|z\|_\alpha = |||f|||_\alpha.$$

We shall continue to refer to the norm so obtained on $C(S) \otimes Y$ as the "lifted" α-norm. To avoid cumbersome notation we use $\|\cdot\|_\alpha$ to refer to both the norm α on $C(S)$ and the "lifted" norm α on $C(S) \otimes Y$. The intention will always be clear from the context. At this stage we shall also need to strengthen our norms to be lattice norms. An easy consequence of the fact that α is a lattice norm on $C(S)$ is the implication

$$|x| \leqslant |y| \Rightarrow \|x\|_\alpha \leqslant \|y\|_\alpha \qquad (x, y \in C(S)).$$

3.8. LEMMA. *Let α be a lattice norm on $C(S)$ and let Y be a normed linear space. The lifted α-norm on $[C(S), \alpha] \otimes Y$ is a reasonable crossnorm.*

Proof. In order to prove the crossnorm property, let $z = x \otimes y$. Then

$$(Jf_z)(s) = \|f_z(s)\| = \|x(s)\, y\| = |x(s)|\ \|y\|.$$

Hence

$$\|z\|_\alpha = |||f_z|||_\alpha = \|Jf_z\|_\alpha = \|\ \|y\| \cdot |x|\ \|_\alpha = \|x\|_\alpha\ \|y\|.$$

For the "reasonable" property of a crossnorm we have to prove that $\|\phi \otimes \psi\| \leqslant \|\phi\|\ \|\psi\|$ whenever $\phi \in [C(S), \alpha]^*$ and $\psi \in Y^*$. Let $z = \sum x_i \otimes y_i$. Then

$$\left|\sum x_i(s)\, \psi(y_i)\right| \leqslant \|\psi\| \left\|\sum x_i(s)\, y_i\right\| = \|\psi\|\ \|f_z(s)\| = \|\psi\|\ (Jf_z)(s).$$

Since α is a lattice norm, we have

$$\left\| \sum \psi(y_i)\, x_i \right\|_\alpha \leqslant \|\psi\| \; \|Jf_z\|_\alpha = \|\psi\| \; \|z\|_\alpha.$$

It follows that

$$|(\phi \otimes \psi) z| = \left| \sum \phi(x_i)\, \psi(y_i) \right| \leqslant \|\phi\| \left\| \sum \psi(y_i)\, x_i \right\|_\alpha \leqslant \|\phi\| \; \|\psi\| \; \|z\|_\alpha. \quad \blacksquare$$

The completion of the normed space $[C(S, Y), \alpha]$ will be denoted by $C_\alpha(S, Y)$.

3.9. LEMMA. *Let α be a lattice norm on $C(S)$, and let Y be a normed linear space. Then there is a natural isometric isomorphism between $C_\alpha(S) \otimes_\alpha Y$ and $C_\alpha(S, Y)$.*

Proof. There are natural embeddings

$$[C(S) \otimes Y, \alpha] \xrightarrow{i} [C(S, Y), \alpha] \xrightarrow{j} C_\alpha(S, Y).$$

The map $k = j \circ i$ has an extension $\bar{k}$ that is an embedding:

$$\bar{k}\colon C(S) \otimes_\alpha Y \to C_\alpha(S, Y).$$

By 3.6, $C_\alpha(S) \otimes_\alpha Y = C(S) \otimes_\alpha Y$. Hence $\bar{k}$ can be regarded as an embedding as follows:

$$\bar{k}\colon C_\alpha(S) \otimes_\alpha Y \to C_\alpha(S, Y).$$

Observe now that if $f \in C(S, Y)$ then

$$|||f|||_\alpha = \|Jf\|_\alpha \leqslant \|2\|_\alpha \; \|Jf\|_\infty = \|2\|_\alpha \; |||f|||_\alpha.$$

Thus any subset of $C(S, Y)$ that is ∞-dense is necessarily α-dense. Now $C(S) \otimes Y$ is ∞-dense in $C(S, Y)$ by Grothendieck's theorem (see, for example, [DU, p. 224] or [LC, p. 9]). Hence $i[C(S) \otimes Y]$ is α-dense in $C(S, Y)$ and therefore also in $C_\alpha(S, Y)$. It follows by an elementary argument that $\bar{k}$ is an isometric isomorphism onto $C_\alpha(S, Y)$. $\blacksquare$

3.10. LEMMA. *Let the notation $\hookrightarrow$ signify a dense embedding between normed linear spaces. We have then*

$$[C(S) \otimes Y, \alpha] \hookrightarrow [C(S, Y), \alpha] \hookrightarrow C_\alpha(S, Y) = C(S) \otimes_\alpha Y = C_\alpha(S) \otimes_\alpha Y.$$

Proof. The embedding on the left is standard. Namely, $\sum x_i \otimes y_i$ is identified with the function $s \mapsto \sum x_i(s)\, y_i$. The next embedding is the natural one of a normed space into its completion. The equalities signify isometric isomorphisms. The first of these is proved in 3.9, and the second is a consequence of 3.4. ▮

It is also easy to prove that

$$[C(S) \otimes Y, \alpha] \subsetneq C_\alpha(S) \otimes Y \subsetneq C_\alpha(S) \otimes_\alpha Y.$$

4. Monotone and Lattice Norms on $C(S \times T)$

If α is a monotone norm on $C(S)$ and if β is any norm on $C(T)$, then α can be lifted to $C(S, C_\beta(T))$ by the technique described in Section 1. It is convenient in this case to indicate both norms in the notation and so the α-norm of f will be denoted by $\|f\|_{\alpha\beta}$. Thus, formally,

$$\|f\|_{\alpha\beta} = \|Jf\|_\alpha, \qquad (Jf)(s) = \|f(s)\|_\beta, \qquad f \in C(S, C_\beta(T)).$$

If α is any norm on $C(S)$ and β is monotone on $C(T)$ then the same mechanism produces a norm $\beta\alpha$. If α and β are both monotone, then $\alpha\beta$ and $\beta\alpha$ can be defined on $C(S \times T)$, although they need not be equal there. If α is a lattice norm then 3.8 guarantees that the $\alpha\beta$-norm is a reasonable crossnorm on $C(S) \otimes C(T)$.

The remainder of this section is concerned with proximinality, and the two theorems given both share the same strategy. In 4.1, the general situation can be described as follows. Suppose that we have a proximinal subspace G in a normed linear space X and an element x in $X \backslash G$ with some additional attractive properties. Then does x possess a best approximation in G which inherits those properties? The technique of 4.1 is to construct a map $L\colon G \to G$ such that $\|x - Lg\| \leqslant \|x - g\|$ for each g in G and such that the range of L contains only members of G which share the desirable properties of x.

In 4.8, proximinality is established for certain subspaces. The technique of proof varies from the usual one of establishing that some subsequence of a minimizing sequence is convergent (because of finite-dimensionality or uniform convexity, for example). With the same notation as above, assume that $[g_i]$ is a minimizing sequence for $x \in X \backslash G$ so that

$$\|x - g_i\| \to \operatorname{dist}(x, G).$$

Again we construct $M\colon G \to G$ such that $\|x - Mg\| \leqslant \|x - g\|$ for each $g \in G$, and such that M is compact in some suitable topology. Then $[Mg_i]$ is also a minimizing sequence but now with a convergent subsequence and the argument proceeds as before. Careful inspection of the two theorems will reveal a close connection between L and M.

In Theorem 4.1 the setting is as follows. We have lattice norms α and β on $C(S)$ and $C(T)$, respectively. It is assumed that $\alpha\beta=\beta\alpha$ on $C(S\times T)$. Subspaces G and H are given in $C_\alpha(S)$ and $C_\beta(T)$. These are assumed to possess continuous proximity maps:

$$A\colon C_\alpha(S)\twoheadrightarrow G, \qquad B\colon C_\beta(T)\twoheadrightarrow H.$$

Now we assume that G and H consist exclusively of continuous functions: $G\subset C(S)$, $H\subset C(T)$. When we wish to associate the α-norm with G we write G_α and when we associate the β-norm with H we write H_β. We also assume that the β- and ∞-norms are equivalent on H. (This is of course true if H is finite-dimensional.) A consequence of this assumption is that if $u\in C(S, H_\beta)$, then $u\in C(S, H_\infty)$ and $u\in C(S, C(T))$.

4.1. THEOREM. *Assume the hypotheses in the preceding paragraph. Let* $f\in C(S\times T)$ *and* $w\in C_\alpha(S)\otimes_{\alpha\beta} H_\beta+G_\alpha\otimes C_\beta(T)$. *Then there exists*

$$\bar{w}\in C_\infty(S)\otimes_\lambda H_\beta+G_\alpha\otimes_\lambda C_\infty(T)$$

satisfying $\|f-\bar{w}\|_{\alpha\beta}\leqslant\|f-w\|_{\alpha\beta}$.

Proof. Write $w=u+v$, where $u\in C_\alpha(S)\otimes_{\alpha\beta} H_\beta$ and $v\in G_\alpha\otimes C_\beta(T)$. Since β is weaker than the ∞-norm, $f\in C(S, C_\beta(T))$. Also, $v\in C(S, C_\beta(T))$. Put $\bar{u}=B\circ(f-v)$ and apply 2.1 to infer that $\bar{u}\in C(S, H_\beta)$ and that for any $z\in C(S, H_\beta)$,

$$|||f-v-\bar{u}|||_\alpha\leqslant|||f-v-z|||_\alpha.$$

Observe that by 3.10, $f-v\in C_\alpha(S)\otimes_{\alpha\beta} C_\beta(T)$. Also by 3.10, $[C(S, H_\beta),.\,\alpha]$ is dense in $C_\alpha(S)\otimes_{\alpha\beta} H_\beta$. Both of these spaces just mentioned are subspaces of $C_\alpha(S)\otimes_{\alpha\beta} C_\beta(T)$. It follows from the density that

$$\|f-v-\bar{u}\|_{\alpha\beta}\leqslant\|f-v-u\|_{\alpha\beta}.$$

This process will now be repeated to replace v by a continuous function. By the remarks prior to the theorem,

$$f-\bar{u}\in C(S\times T)\subset C(T, C_\alpha(S)).$$

Hence we can define $\bar{v}=A\circ(f-\bar{u})$. By 2.1, $\bar{v}\in C(T, G_\alpha)$ and for any $z\in C(T, G_\alpha)$,

$$|||f-\bar{u}-\bar{v}|||_\beta\leqslant|||f-\bar{u}-z|||_\beta.$$

By 3.10, $C(T, G_\alpha)$ is β-dense in $C_\beta(T) \otimes_{\alpha\beta} G_\alpha$. Also, $v \in C_\beta(T) \otimes_{\alpha\beta} G_\alpha$. Hence

$$\begin{aligned} \|f - \bar{u} - \bar{v}\|_{\alpha\beta} &= \|f - \bar{u} - \bar{v}\|_{\beta\alpha} \\ &\leqslant \|f - \bar{u} - v\|_{\beta\alpha} \\ &= \|f - \bar{u} - v\|_{\alpha\beta} \\ &\leqslant \|f - u - v\|_{\alpha\beta}. \end{aligned}$$

The proof is complete with $\bar{w} = \bar{u} + \bar{v}$. ▮

4.2. Corollary. *Let G and H be finite-dimensional subspaces in $C(S)$ and $C(T)$, respectively. Let μ and ν be σ-finite, positive, Borel measures whose supports are S and T, respectively. Then the subspace*

$$W = G \otimes C(T) + C(S) \otimes H$$

is L_p-Chebyshev in $C(S \times T)$. (Here $1 < p < \infty$.)

Proof. In the preceding theorem, let α and β be the L_p-norms on $C(S)$ and $C(T)$, respectively. Then $C_\alpha(S) = L_p(S, \mu)$ and $C_\beta(T) = L_p(T, \nu)$. Also by the Fubini theorem, $\|f\|_{\alpha\beta} = \|f\|_{\beta\alpha}$ for all $f \in C(S \times T)$. Since G and H are finite-dimensional, the subspace

$$W' = G \otimes L_p(T) + L_p(S) \otimes H$$

is closed in $L_p(S \times T)$ by 11.2 in [LC]. If $f \in C(S \times T)$ then f has a best L_p-approximation w' in W' because $L_p(S \times T)$ is a uniformly convex Banach space and W' is closed. Since G and H are finite-dimensional subspaces in $L_p(S)$ and $L_p(T)$, there exist proximity maps fulfilling the hypotheses of the preceding theorem. Hence, by that theorem, there exists $w \in W$ for which

$$\|f - w\|_{\alpha\beta} \leqslant \|f - w'\|_{\alpha\beta}.$$

Since the $\alpha\beta$-norm is the L_p-norm on $C(S \times T)$, the function w is the best approximation sought, and in fact is w' by strict convexity. ▮

4.3. Lemma. *Let α and β be lattice norms on $C(S)$ and $C(T)$, respectively. Assume that $\alpha\beta = \beta\alpha$. Then the crossnorm $\alpha\beta$ is uniform on $[C(S), \alpha] \otimes [C(T), \beta]$.*

Proof. Let A and B be bounded operators on $[C(S), \alpha]$ and $[C(T), \beta]$, respectively. Let $x_i \in C(S)$ and $y_i \in C(T)$. We want to prove that

$$\left\| \sum Ax_i \otimes By_i \right\|_{\alpha\beta} \leqslant \|A\| \, \|B\| \left\| \sum x_i \otimes y_i \right\|_{\alpha\beta}.$$

We have

$$J\left(\sum Ax_i \otimes By_i\right)(s) = \left\| \sum (Ax_i)(s) \cdot By_i \right\|_\beta \leqslant \|B\| \left\| \sum (Ax_i)(s) \cdot y_i \right\|_\beta$$
$$= \|B\| \, J\left(\sum Ax_i \otimes y_i\right)(s).$$

Hence, by the monotonicity of α,

$$\left\| \sum Ax_i \otimes By_i \right\|_{\alpha\beta} \leqslant \|B\| \left\| \sum Ax_i \otimes y_i \right\|_{\alpha\beta}.$$

By interchanging α and β, the same argument leads to

$$\left\| \sum Ax_i \otimes By_i \right\|_{\beta\alpha} \leqslant \|A\| \left\| \sum x_i \otimes By_i \right\|_{\beta\alpha}.$$

Taking $B = I$ in this last inequality and using $\alpha\beta = \beta\alpha$, we have

$$\left\| \sum Ax_i \otimes y_i \right\|_{\alpha\beta} \leqslant \|A\| \left\| \sum x_i \otimes y_i \right\|_{\alpha\beta}. \quad \blacksquare$$

4.4. Lemma. *The $\infty\beta$-norm is uniform on $[C(S), \infty] \otimes [C(T), \beta]$.*

Proof.

$$\left\| \sum Ax_i \otimes By_i \right\|_{\infty\beta} = \sup_s \left\| \sum (Ax_i)(s)\, By_i \right\|_\beta$$
$$\leqslant \|B\| \sup_s \left\| \sum (Ax_i)(s)\, y_i \right\|_\beta$$
$$= \|B\| \sup_s \sup_\psi \left| \sum (Ax_i)(s)\, \psi(y_i) \right|$$
$$\leqslant \|B\| \sup_\psi \left\| \sum \psi(y_i)\, Ax_i \right\|_\infty$$
$$\leqslant \|A\| \, \|B\| \sup_\psi \left\| \sum \psi(y_i)\, x_i \right\|_\infty$$
$$= \|A\| \, \|B\| \, \lambda\left(\sum x_i \otimes y_i\right)$$
$$\leqslant \|A\| \, \|B\| \left\| \sum x_i \otimes y_i \right\|_{\infty\beta}.$$

In the last step we appeal to 3.8, which asserts that $\infty\beta$ is a reasonable crossnorm; hence it dominates the λ-norm. ∎

4.5. LEMMA. *Let G and H be finite-dimensional subspaces in $C(S)$ and $C(T)$, respectively. Let $U = G \otimes C(T)$ and $V = C(S) \otimes H$. Let α and β be lattice norms on $C(S)$ and $C(T)$, respectively. Assume that $\alpha\beta = \beta\alpha$. Then there is a constant c such that each member w of $U + V$ has a representation $w = u + v$ in which $u \in U$, $v \in V$, and $\|u\|_{\alpha\beta} + \|v\|_{\alpha\beta} \leqslant c\,\|w\|_{\alpha\beta}$.*

Proof. By 4.3 and by [LC, 11.2], the subspace

$$W' = G \otimes C_\beta(T) + C_\alpha(S) \otimes H$$

is $\alpha\beta$-closed in $C_{\alpha\beta}(S \times T)$. Hence by [LC, 11.3] there is a constant c such that each element w in W' has a representation $w = u' + v'$ in which

$$u' \in G \otimes C_\beta(T), \qquad v' \in C_\alpha(S) \otimes H, \qquad \|u'\|_{\alpha\beta} + \|v'\|_{\alpha\beta} \leqslant c\,\|w\|_{\alpha\beta}. \quad (2)$$

Now let w be an element of W. Then it belongs to W'. Select u' and v' such that $w = u' + v'$ and such that (2) is true. By 3.7, $u' \in U$ and $v' \in V$. ∎

4.6. LEMMA. *Let α be a lattice norm on $C(S)$, β any norm on $C(T)$. Let H be a finite-dimensional subspace of $C(T)$. Then there is a constant k_1 such that for $v \in C(S) \otimes H$,*

$$\|v\|_{\infty\alpha} \leqslant k_1\,\|v\|_{\alpha\beta}.$$

Proof. Select a biorthonormal system $\{h_i, \psi_i\}_1^m$ for $[H, \beta]$. Let $v \in C(S) \otimes H$, and write $v = \sum_{i=1}^m x_i \otimes h_i$. Then $v_s = \sum x_i(s)\,h_i$, and so

$$|x_i(s)| \leqslant |\langle v_s, \psi_i\rangle| \leqslant \|v_s\|_\beta\,\|\psi_i\| = \|v_s\|_\beta.$$

Since α is a lattice norm,

$$\|x_i\|_\alpha \leqslant \|v\|_{\alpha\beta}.$$

Now we can make the estimate

$$\|v^t\|_\alpha = \left\|\sum h_i(t)\,x_i\right\|_\alpha \leqslant \sum |h_i(t)|\,\|x_i\|_\alpha \leqslant \sum \|h_i\|_\infty\,\|v\|_{\alpha\beta}.$$

By taking a supremum in t we arrive at

$$\|v\|_{\infty\alpha} \leqslant \|v\|_{\alpha\beta} \sum \|h_i\|_\infty. \quad ∎$$

4.7. LEMMA. *Let α be a norm on $C(S)$. Let G be a finite-dimensional* *subspace of $C(S)$. Let $A: C(S) \twoheadrightarrow G$ be a ∞-continuous, α-proximity map. Define $(\bar{A}z)(s, t) = (Az^t)(s)$ for $z \in C(S \times T)$. Then there is a constant k_3 such that for all z,*

$$\|\bar{A}z\|_\infty \leqslant k_3 \|z\|_{\infty\alpha}.$$

Proof. Let $\{g_i, \phi_i\}_1^n$ be a biorthonormal system for $[G, \alpha]$. Let $z \in C(S \times T)$ and put $v = \bar{A}z$. Then for appropriate $y_i \in C(T)$, $v = \sum_{i=1}^n g_i \otimes y_i$. Hence $v^t = Az^t$ and $v^t = \sum_{i=1}^n y_i(t)\, g_i$. Hence

$$|y_i(t)| = |\langle v^t, \phi_i \rangle| \leqslant \|v^t\|_\alpha \|\phi_i\| = \|v^t\|_\alpha \leqslant 2 \|z^t\|_\alpha.$$

It follows that $\|y_i\|_\infty \leqslant 2 \|z\|_{\infty\alpha}$. Hence

$$\|v\|_\infty \leqslant \sum_{i=1}^n \|g_i\|_\infty \|y_i\|_\infty \leqslant \|z\|_{\infty\alpha} 2 \sum_{i=1}^n \|g_i\|_\infty. \quad \blacksquare$$

4.8. THEOREM. *Let α and B be lattice norms on $C(S)$ and $C(T)$, respectively, and assume that $\alpha\beta = \beta\alpha$. Let G be a finite-dimensional subspace of $C(S)$ having an ∞-continuous α-proximity map. Let H be a finite-dimensional subspace of $C(T)$ having an ∞-Lipschitz, β-proximity map. Then $C(S) \otimes H + G \otimes C(T)$ is $\alpha\beta$-proximinal in $C(S \times T)$.*

Proof. Let $A: C(S) \twoheadrightarrow G$ and $B: C(T) \twoheadrightarrow H$ be the proximity maps whose existence is hypothesized. Extend these to $C(S \times T)$ by defining, for $z \in C(S \times T)$,

$$(\bar{A}z)(s, t) = (Az^t)(s) \qquad (\bar{B}z)(s, t) = (Bz_s)(t).$$

The ranges of $\bar{A}$ and $\bar{B}$ are the subspaces

$$U = G \otimes C(T), \qquad V = C(S) \otimes H,$$

respectively. Also, $\bar{A}$ and $\bar{B}$ are $\alpha\beta$-proximity maps, by 2.1.

Now fix $z \in C(S \times T)$ and define $\Gamma: U \to V$ by the equation $\Gamma(u) = \bar{B}(z - u)$. By [LC, 2.23], Γ is ∞-compact from U to V. Let w_k be a minimizing sequence in $U + V$ for the approximation of z. Thus

$$\|z - w_k\|_{\alpha\beta} \to \operatorname{dist}_{\alpha\beta}(z, U + V).$$

Without loss of generality we can assume $\|w_k\|_{\alpha\beta} \leqslant 2 \|z\|_{\alpha\beta}$. By 4.5, each w_k has a representation $w_k = u_k + v_k$ in which $u_k \in U$, $v_k \in V$, and $\|u_k\|_{\alpha\beta} + \|v_k\|_{\alpha\beta} \leqslant C$.

Define $u'_k = \bar{A}(z - v_k)$. Then by 4.7 and 4.6,

$$\begin{aligned} \|u'_k\|_\infty = \|\bar{A}(z - v_k)\|_\infty &\leqslant k_3 \|z - v_k\|_{\infty\alpha} \leqslant k_3 \|z\|_{\infty\alpha} + k_3 \|v_k\|_{\infty\alpha} \\ &\leqslant k_3 \|z\|_\infty + k_3 k_1 \|v_k\|_{\alpha\beta}. \end{aligned}$$

Thus the sequence $\|u'_k\|_\infty$ is bounded. Define $v'_k = \bar{B}(z - u'_k) = \Gamma(u'_k)$. Since $\|u'_k\|_\infty$ is bounded and Γ is ∞-compact, the sequence $[v'_k]$ lies in an ∞-compact subset of V. Let v be an ∞-cluster point of $[v'_k]$, and define $u = \bar{A}(z - v)$. Now write

$$\|z - u - v\|_{\alpha\beta} \leqslant \|z - u'_k - v\|_{\alpha\beta} \leqslant \|z - u'_k - v'_k\|_{\alpha\beta} + \|v'_k - v\|_{\alpha\beta}.$$

If k runs through a suitable sequence of integers, $\|v'_k - v\|_\infty$ will converge to zero. Hence $\|v'_k - v\|_{\alpha\beta}$ will converge to zero. Since

$$\|z - u'_k - v'_k\|_{\alpha\beta} \leqslant \|z - u'_k - v_k\|_{\alpha\beta} \leqslant \|z - w_k\|_{\alpha\beta}$$

we see that $u + v$ is a best $\alpha\beta$-approximation of z in $U + V$. ∎

Example. Let S be a disconnected space with at least n components. Then S can be expressed as the union of a disjoint family of n open and closed sets, say $S = \bigcup_{i=1}^n S_i$. On each space $C(S_i)$ we consider the subspace $\Pi_0(S_i)$ of constant functions. By 7.15 of [LC] there exist L_1-proximity maps $A_i \colon C(S_i) \twoheadrightarrow \Pi_0(S_i)$ such that

(i) A_i is monotone: $A_i x \geqslant A_i y$ if $x \geqslant y$;

(ii) $A_i(x + \lambda) = A_i x + \lambda$ if $\lambda \in R$;

(iii) $\|A_i x - A_i y\|_\infty \leqslant \|x - y\|_\infty$.

These assertions apply to arbitrary x and y in $C(S_i)$.

In $C(S)$, we define G to be the n-dimensional subspace of piecewise constant functions:

$$G = \{x \in C(S) : x \mid S_i \in \Pi_0(S_i) \text{ for } 1 \leqslant i \leqslant n\}.$$

Define $A \colon C(S) \twoheadrightarrow G$ by piecing together the maps A_i so that $(Ax) \mid S_i = A_i(x \mid S_i)$. Elementary calculations now show that A is an L_1-proximity map of $C(S)$ onto G and that

$$\|Ax - Ay\|_\infty \leqslant \|x - y\|_\infty, \qquad x, y \in C(S).$$

The subspace G has therefore a ∞-Lipschitz, β-proximity map, if β denotes the L_1-norm. Such a subspace satisfies the hypotheses placed on G or H in Theorem 5.

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